The Review of Modern Logic
Volume 10 Numbers 3 & 4 (March 2005–May 2007) [Issue 32], pp. 91–111.

ARE THE NATURAL NUMBERS JUST ANY PROGRESSION? PEANO, RUSSELL, AND QUINE

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Are the natural numbers just any progression? It is widely held that Peano and Quine say yes, Russell no. For Russell criticizes Peano, and Peano and Quine criticize Russell.[1] The paper has four parts. (1) I describe Peano's theory as Russell understands it and as I think it is. (2) I describe Russell's criticism. (3) I extend Russell's criticism to odd counting procedures. (4) I discuss Quine's objections to Russell. I conclude that while it is not in the least controversial that infinitely many definitions of numbers and counting procedures are possible, Russell is right and my extension of Russell is right.[2]

1. Peano's Theory

Russell praises Peano's theory for being correct as far as it goes. He believes that it is an adequate theory of purely arithmetical equations such as $2 + 2 = 4$. He assigns it permanent value in the history of mathematics because it reduces arithmetic, and by extension all mathematics, to three undefined terms and five axioms.[3]

Received 12/27/2006.

[1][Russell 1903, 124–8], [Russell 1919, 5ff.], [Russell 1948, 236–7]. Peano returns the favor and criticizes Russell's definition; see [Russell 1903, 115]. On whether Peano invented the definition of number as a class of classes before Russell, see [Rodríguez-Consuegra 1991, 156–7] and [Kennedy 1974, 401–2].

[2]The seed of this paper was my e-mail discussion of Russell and Peano with Gregory Landini, Raymond Perkins, Torkel Franzén, Daniel Kervick, and Donald Stahl in Russell-l, the International Forum for Bertrand Russell Studies, `http://mailman.mcmaster.ca/mailman/listinfo/russell-l`, an "unadvertised" mailing list, from late 1999 to early 2000. I thank the group moderator for kindly granting me permission to make use of material from that discussion, which is in the Russell-l archives.

[3][Russell 1919, 5, 6–7]. Peano's original paper of 1889 uses four undefined terms and nine axioms [Kennedy 1980, 26]. Peano's axioms remain valuable for modeling subclasses of arithmetical truths not subject to Kurt Gödel's incompleteness theorem [Quine 1987, 16]; see [Kaye, 42–3]. Nearly all of the over 230 papers on Peano since 1940 are formal work on the axioms.

Concerning Peano's three undefined terms, "number," "0," and "the successor of," Russell takes "number" to be variously interpretable as the class of items the natural numbers are, "0" to be variously interpretable as the specific natural number 0, and "succession" to be variously interpretable as the relation each natural number has to the next natural number after it.[4]

Russell seems overgenerous to Peano at this stage. The numbers are simply 0 and the successors. To identify the numbers is simply to identify 0 and the successors. Thus "number" can be defined as "0 or a successor." Alternatively, "0" can be defined as "the only number which has no successor".[5] Or thirdly, "successor" can be defined as "number other than 0." Thus any one of Peano's three undefined terms can be defined using the other two. The question which term to define depends on which pair of undefined terms best illuminates arithmetic.[6]

Russell describes Peano's five axioms as containing only the three undefined terms plus logical terms:

(1) 0 is a number.
(2) The successor of any number is a number.
(3) No two numbers have the same successor.
(4) 0 is not the successor of any number.
(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.

[Russell 1919, 5–6], [Russell 1903, 125]

Russell quotes Peano to the apparent effect that for Peano, numbers are a mere abstraction of formal properties. That is, it does not matter what numbers are or what we take them to be, so long as the axioms are satisfied. Thus for Russell, Peano's theory is formalist or structuralist [Russell 1903, 125]. If it is a definition, it is implicit or contextual.

Russell seems mistaken about Peano's intent, and Russell's quote of Peano's "formalism" seems misleading. [Kennedy 1974], [Gillies], [Segre], and [Rodríguez-Consuegra 1991] agree that for Peano, (1) number is indefinable, (2) the axioms are not definitions but axioms, and do not define number or anything else, but only state the properties necessary and sufficient for arithmetical equations to have their proper truth-values, and (3) admittedly there are infinitely many series of

[4]From here on, I shall omit the adjective "natural," using "number" to mean 0 and the positive integers.

[5]Russell seems well aware of this [Russell 1903, 125].

[6]I favor the Frege-Russell approach of using "0" and "successor" to define number.

things which satisfy the axioms as well as the numbers do, but which are not the numbers, but merely various progressions. After a lifetime of flip-flopping on the question, Peano's final view was that axioms might be considered definitional. But he qualified this by saying "in a certain fashion," and he was talking about axioms in geometry, not in arithmetic.[7]

Of course, insofar as it is theoretically desirable to define number if possible, and to leave no stone unturned, Russell is still right to consider whether number can be defined in terms of Peano's axioms and to criticize the suggestion, whether it was what Peano intended or not. In this sense, Russell's criticism remains as valid as ever. And it would seem to apply to Dedekind, if not to Peano [Dedekind, 62–80].[8]

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[7]As translated in [Kennedy 1974, 404], citing [Peano, 244], where Kennedy translates the equivocation as "in a certain way." Kennedy chronicles Peano's thinking on whether number can be defined [Kennedy 1974, 395–406]. Peano finds the concept of number too fundamental to define, and can only axiomatize it [Kennedy 1974, 405–6]. Peano finally accepts definition by abstraction only in 1921, and even then Peano is "still equivocating" [Kennedy 1974, 404]. Segre agrees, saying Peano "believed that basic entities, such as 'zero', 'number', and 'successor' *cannot* be defined" [Segre, 203] (see also [Segre, 286], [Segre, 291] citing [Kennedy 1963, 263]). Gillies agrees, too [Gillies, 66–7] (see [Gillies, 68]). So does [Rodríguez-Consuegra 1991, 107] citing [Gillies]; but [Rodríguez-Consuegra 1991, 106–8] also indicates that Peano often flip-flopped on whether numbers can be defined. Both Peano and Russell seem well aware that finding five axiomatic properties true of two progressions is a far cry from finding that they are the same progression.

[8]Segre says,

> ...Dedekind practically formulated Peano's axioms one year before Peano, but as Peano himself modestly points out, there is a slight difference between Dedekind, who defines number according to the conditions it satisfies, and Peano, who does not want to define number and only states its basic properties. [Segre, 292]

Rodríguez-Consuegra argues that Russell's "criticisms of Peano and Dedekind have a common idea: they both really defined, not numbers, but *progressions*" [Rodríguez-Consuegra 1991, 172] (see [Rodríguez-Consuegra 1991, 90n]; see also [Segre, 294 n. 24]).

Wang says, "It is rather well-known, through Peano's own acknowledgment, that Peano borrowed his axioms from Dedekind" [Wang, 145]. But I agree with Rodríguez-Consuegra that Peano probably developed the axioms independently, since Peano's acknowledgment is only general, and elsewhere Peano specifically denies taking the axioms from Dedekind [Rodríguez-Consuegra 1991, 106] (see also [Segre, 292, 292 n. 21], [Quine 1987, 16]).

2. Russell's Criticism of Peano

Russell's criticism is that Peano's definition implies wrong counts of things, e.g., sheep. The criticism is deep. Counting is basic to all science, not just to the logical analysis of arithmetic.

Russell poses his criticism as a dilemma. Either number-terms already have independent, "constant" meanings or they do not. If they do, then there is no need to define numbers by abstraction. If they do not, then definition by abstraction will not provide them with "constant" meanings, but only with "variable" meanings. So to speak, the axioms will function as an implicit or contextual definition of all three undefined terms at once (or two, if we define one term in terms of the other two). The three undefined terms will function as variables in the axioms. Thus there will be no existence-proof that numbers exist, since only a class of trios will be defined, not the particular trio of entities which number, 0, and successor actually are. But number-terms need to have constant meanings, if we are to be able to count things. For determinate counts require determinate denotative meanings for our number-terms [Russell 1903, 126–7].

Russell's criticism has two logical stages. (1) We show that Peano's undefined terms can be interpreted in different ways that satisfy Peano's axioms. (2) We show that on many of these interpretations, our counts of things will conflict with our ordinary and correct counts, and thus will be wrong. Stage (2) is the actual criticism, since Peano would accept stage (1).

I shall use stars (asterisks) to mark Russell's odd arithmetics. For example, Russell says we can take "1" to mean 100, and "number" to mean numbers from 100 on. Here "successor" is to mean what it ordinarily does, that is, the result of adding one. Call this odd interpretation of arithmetic odd* (pronounced "odd star"). Or we can take "0" to mean 0, "number" to mean even numbers, and "successor" to mean the result of adding two. Call this second odd interpretation of arithmetic odd** ("odd double star"). Both odd arithmetics satisfy Peano's axioms. But in odd*, "two sheep" means 101 sheep, and in odd**, "two sheep" means four sheep [Russell 1919, 9]. Russell says that Peano's axioms really define the class of progressions. Things compose a progression if they can be placed in one-one correspondence with the numbers.[9] "The progression need not be composed of numbers: it

[9]Of course, defining the numbers as any progression is circular, if being a progression is defined as being in one-one correspondence to the numbers. But Russell defines both progressions and numbers in terms of logic and class theory, following Frege.

may be composed of points in space, or moments of time, or any other terms of which there is an infinite supply" ([Russell 1919, 8–9]. Russell even allows a number-term to denote for Peano an ordinary object such as Cleopatra's Needle [Russell 1919, 9]. For any progression will satisfy Peano's axioms and thereby all the true equations of pure arithmetic. Russell says:

> For instance, let "0" mean what we commonly call "1," and let "number" mean what we commonly call "number other than 0"[, and let "successor" mean the result of adding one]; then all the five [axioms] are still true, and all arithmetic can be proved, though every formula will have an unexpected meaning. "2" will mean what we usually call "3," but "2 + 2" will not mean "3 + 3"; it will mean "3 + 2," and "2 + 2 = 4" will mean what we usually express by "3 + 2 = 5."
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> [Russell 1948, 237]

Call this third odd interpretation of arithmetic odd*** ("odd triple star"). Evidently Russell takes the "2" embedded in the function expression "x + 2" as meaning what it ordinarily does. That is, for Russell odd*** "2 + 2 = 4" has two occurrences of "2," the first being odd*** and the second being ordinary. The natural suggestion is that we can rewrite odd*** "2 + 2" more perspicuously as "2*** + 2," so that "2*** + 2 = 4***" means the same as "3 + 2 = 5." It is hard to see what else Russell can have in mind here. If this suggestion is correct, then Russell implicitly uses ordinary language, including ordinary arithmetical language, as a metalanguage in which he defines odd*** as an object language which is syntactically the same as ordinary arithmetic,[10] but semantically different in the denotations of its terms. That is, his odd*** interpretation of number-terms results in the odd*** object language about numbers.

Russell notes in effect that "2*** + 2 = 4***" is true in odd*** if and only if "2 + 2 = 4" is true in ordinary arithmetic. In Quinean terms, the reason for odd***'s preservation of the equation's truth-value is that the move from 2 to 2*** and the move from 4 to 4*** are mutually "compensatory adjustments in the translation of other words" [Quine 1969, 29].[11] But Russell in effect notes that if we count sheep in odd***, 4*** sheep are five sheep, not four sheep. Thus odd*** fails

[10]Russell does not use stars or hash marks.

[11]Compare Wittgenstein [Wittgenstein, I § 136] on mutually compensatory adjustments involving counting.

to preserve the truth-value of ordinary empirical counting statements. That is because there is no compensatory adjustment to counting.

Of course, Peano escapes between the horns of Russell's dilemma, if Peano is not trying to define number in the first place. But Dedekind and Quine would seem to be caught.[12]

3. Russell's Criticism Extended to Odd Counting

Russell's odd*** is not a purely odd arithmetic. For "2*** + 2 = 4***" is a hybrid of odd*** and ordinary terms, the latter being "+," "2," and "=."

Let a *purely* odd arithmetic be an arithmetic in which every number-term and every arithmetical operator-term (plus, minus, and so on) has an odd interpretation, and let a *totally* odd arithmetic be a purely odd arithmetic which also has an odd interpretation of counting.[13] We can stipulate infinitely many totally odd arithmetics which preserve pure arithmetic but fail to preserve ordinary counting. But we can also stipulate infinitely many totally odd arithmetics which *do* preserve ordinary counting, simply by stipulating compensatory adjustments in

[12]For Russell, the deeper basis of Peano's dilemma is Peano's confusion of variable and constant. Zaitsev says:

> the variable x in a functional expression may play the role of the variable proper (in the modern sense of the word) and an arbitrary value of it .... Peano's vagueness on this point was criticized by Russell[, who] indicated that "formal and material implication are combined by Peano into one primitive idea, whereas they ought to be kept separately. [...] The complication introduced at this point arises from the nature of the variable, a point which Peano, though he has done very much to show its importance, appears not to have himself sufficiently considered" .... When a theory is constructed without any clear distinction between sign and signification, as does Peano, the confusion between variables and constants is inevitable. I think that this is the reason why Peano never answered Frege's, Vailati's, Russell's or Jourdain's criticisms. To distinguish properly between variables and constants would entail much trouble in the general framework of his logical conception.
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> [Zaitsev, 371–3], quoting [Russell 1903, 28]

But while Russell may perceive this as the deeper basis, Rodríguez-Consuegra argues that Peano is actually quite clear on variables [Rodríguez-Consuegra 1991, 102], and I am inclined to agree.

[13]For easy exposition, let the equals sign be the purely logical term of identity, and always be interpreted normally, though this sign too can be interpreted oddly.

their counting procedures. I shall use hash marks to indicate my own purely odd and totally odd arithmetics and their expressions.[14]

Let odd### ("odd triple hash") be the same as Russell's odd***, except that the addition expression "x +### y" is to mean "$(x + y) - 1$." Thus "2### +### 2### = 4###" is a purely odd### equation, if we treat the equals sign as the logical identity sign. Since converting the second occurrence of "2" in "2 + 2" from ordinary to odd### adds one, I am simply subtracting one from the ordinary addition function for a net change of zero. Odd### arithmetic preserves the pure equation $2 + 2 = 4$ as well as odd*** does. Ordinary, odd###, and odd*** pure arithmeticians all would agree that "two plus two equals four" is true, where "2### +### 2### = 4###" means the same as ordinary "$(3 + 3) - 1 = 5$." But once again counting fails to be preserved. In ordinary English without hash marks, where ordinary counters would say, "There are five sheep," odd### counters would say, "There are four sheep." That is because odd###

[14]Evidently inspired by Wittgenstein, Gasking had the idea of compensating for odd arithmetics by using odd counting procedures before I did [Gasking]. Gasking suggests we can also compensate for odd arithmetics by adjusting our conventional laws about counting, e.g. about objects disappearing or coalescing, or about measurement, e.g., about the expansion or contraction of objects. This is as opposed to laws of nature, which cannot be adjusted; but he adds that if the unadjustable natural laws were very different from what they are, e.g. with objects really vanishing or really expanding when we count or measure, we might find it "enormously convenient" to alter our mathematics. For although (or better, because) our mathematics is incorrigible, it is not about the world [Gasking] (compare [Wittgenstein, I §§ 5, 139]). Castañeda is right that all the odd arithmetics and odd counting procedures in my own paper are trivially odd in that they do not contradict the propositions ordinary arithmeticians and ordinary counters assert, e.g., they do not really say that $1 + 1 = 3$. Castañeda thinks that "more exciting" odd arithmetics would be like "non-Euclidean geometries" [Castañeda, 409]. This is a slip. The non-Euclidean geometries are trivially intertranslatable with Euclid [Nagel, 235–41]. They admit that given a *Euclidean* line on a *Euclidean* plane, there is exactly one *Euclidean* line through a *Euclidean* point on the given plane but not on the given line. That is, they affirm their respective translations of that as true. Russell himself is very clear on the arithmetic-geometry comparison, and even cites the non-Euclidean geometries as an example that makes his criticism of Peano clear [Russell 1927a, 4–5]. Russell explains that in both arithmetic and geometry, all sorts of unimportant interpretations may be offered in theory, but only one interpretation may be important for empirical applications [Russell 1927a, 4–5]. Castañeda says correctly, "Peano's postulates alone do not constitute a queer arithmetic, but only an incomplete formulation of arithmetic. Those postulates do not tell us how numbers are used in counting and measuring" [Castañeda, 413]. Perhaps then Castañeda would find our odd arithmetics trivial because he finds Russell obviously right. But other philosophers, such as Quine, find Peano obviously right.

is not totally odd. It uses ordinary one-one correspondence for counting sheep, so that there is no compensatory adjustment in its counting procedure for the oddness of its number-terms.

In pure arithmetic, we cannot tell who is speaking in ordinary arithmetic and who is speaking in odd###. But we can easily tell who is counting sheep ordinarily as opposed to oddly###. Clearly, our odd arithmetics are not yet as odd as they could be. It seems we must stipulate odd counting procedures, if our approach is to rise to the level of Quine's theses of referential inscrutability and translational indeterminacy.

Counting is ordinarily understood as placing things into one-one correspondence with the numbers. Where all our ducks are in a row, we teach children to point to the first duck and say "one," and so on. But if we are to admit odd counting procedures, we need a wider concept.

Let a counting procedure be any specifiable way of placing things into correspondence with numbers in conformance to Peano's axioms. In a totally odd arithmetic, a compensatory adjustment in counting procedure will consist of adding some number to or subtracting some number from the ordinary count, so that the odd counter will say "There are four ducks" if and only if an ordinary counter would, despite the odd counter's odd interpretation of numbers. We may conceive of "adding to the ordinary count" as "repeated counting of the first duck." We may also conceive of "subtracting from the ordinary count" as "counting some initial ducks as a set instead of individually," or more simply as "skipping over some initial ducks." But such conceptions are not necessary, and sometimes they are not even possible. When there are zero ducks, there *are* no initial ducks to count repeatedly, bundle into a set, or skip over, but we can still easily add a number to or subtract a number from an ordinary count of zero.

Let odd#### be the same as odd### except that an odd#### count is the ordinary count minus one. Thus my compensatory adjustment is once again to subtract one for a net change of zero. But this time I am subtracting one from the count of ducks instead of from an addition in an equation. Ordinary and odd#### counters will both truly say "There are four ducks" where there are four ducks. Thus in odd####, Russell's criticism of Peano vanishes. In odd####, our compensatory extension of Russell's criticism paradoxically destroys that very criticism.

Our philosophical achievement is to place Peano's structuralism at the depth of Quine's. If Russell's criticism has indeed vanished, then there is no saying what the numbers are. There will be no observable difference between ordinary and odd#### counters, assuming the

odd#### counters invent and apply odd#### in their heads and do not tell anyone. The only difference is that the ordinary counter internally means that there are four ducks, while the odd#### counter internally means that there are five ducks minus one. The two meanings are distinct only in reason, but that entails they are different—just like Quine's rabbits, undetached rabbit parts, and temporal rabbit slices, which are distinct only in reason. But this achievement is only in odd#### and in odd arithmetics like it. Odd*, odd**, odd***, and odd### remain valid counterexamples to Peano's theory as Russell understands it.[15]

What our scholarly achievement is depends on how broadly to construe Peano and Russell. Arguably, odd#### is logically implicit in their thoughts. But if we go narrowly by the letter of their writings, which discuss only ordinary one-one correspondence, surely they did not think of the possibility of odd counting procedures. Otherwise, surely Russell would have seen his criticism as futile in cases like odd####. In my opinion, Russell refutes Peano (or at least Dedekind), and his refutation can be extended to odd counting procedures. But if we extend it to odd counting procedures with compensatory adjustments, then in those cases Russell's refutation refutes itself, and we achieve a Quinean structuralism.

That does not mean that Quine wins due to odd####, or that Russell wins due to odd###. The deeper issue is philosophical. Namely, Russell would reject Quine's verificationist thesis that it is meaningless to say what numbers really are, while Quine would reject Russell's broadly speaking phenomenological thesis that odd counts are empirically given as wrong.[16] I agree with Russell against Quine, since like

[15]Rodríguez-Consuegra argues that due to the later Russell's general structuralist tendencies, the later Russell is surely committed to accepting Peano's structural definition of numbers [Rodríguez-Consuegra 1996, 226]. Rodríguez-Consuegra overlooks that in 1948 Russell uses empirical data of counting to refute Peano as usual [Russell 1948, 237]. In fact, the later Russell holds that if everything is a purely logical structure, then "there is a certain air of taking in each other's washing about the whole business" [Russell 1927b, 153], and "the physical world will run through our fingers like a jelly-fish" [Russell 1927a, 319]. As Quine knows, Russell never extends his structuralism to what is sensibly presented [Quine 1980–1, 20]; this implicitly includes empirical counting.

[16]Russell is well aware of this dialectic. Again, the first horn of the dilemma he poses is that if number-terms already have independent, "constant" meanings, then there is no need to define them by abstraction, i.e., structurally [Russell 1903, 126]. And Quine is well aware that Russell's criticism of Peano poses a challenge to the theses of referential inscrutability and translational indeterminacy [Quine 1969, 44–5]. Quine rejects Russell's requirement that numbers have determinate empirical

many I find verificationism unverifiable. But due to the generality of Quine's theses of referential inscrutability and translational indeterminacy, Russell's criticism of Peano loses its independent interest.

Ayer carries Russell's warfare into Quine's camp. Ayer argues that even from within the perspective of verificationism, structuralism is wrong:

> [I]t is a mistake to draw a distinction between the structure and the content of people's sensations—such that the structure alone is accessible to the observation of others, the content inaccessible. For if the contents of other people's sensations really were inaccessible to my observation, then I could never say anything about them. But, in fact, I do make significant statements about them; and that is because I define them, and the relations between them, in terms of what I myself can observe.
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> [Ayer, 132][17]

But we can also observe the number of spots on a card. Thus Ayer's argument applies to the content and structure of numbers as well. Quine himself urges the analogy between systematic color inversions and systematic number inversions [Quine 1969, 50] (see [Russell 1927a, 224]). That Quine would reject Ayer's individual statement meanings in favor of meaning holism does not matter, if there can be holist contents as well as structures. More subtly, Ayer might be considered to support Quine against Russell. If we substitute applications for contents, then Ayer might be confirming Quine's view that "Always, if the structure is there, the applications will fall into place" [Quine 1969, 44]. Of course, Ayer speaks of contents of sense-experiences, which is closer to Russell than to Quine. But these contents must be either publicly observable or unverifiable for Ayer, which is closer to Quine than to Russell.

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applications on the ground that ostensions themselves can be multiply interpreted [Quine 1969, 44]. But Quine's criticism can succeed (if at all) only in behavioristic philosophies like his own.

[17]Arguably, Quine agrees when he admits prelinguistic quality spaces. Quine argues that without prelinguistic resemblances and dissimilarities, verbal training would be impossible [Quine 1960, 83–4]. Quine adds that in psychological testing, "a criterion can never attest to prelinguistic quality spaces except as they are uniform from child to child" [Quine 1960, 84]. That is because for Quine, everything in science must be objective for purposes of communication and of evidence. But then why cannot empirically given numbers of things also be contents as opposed to structures? Even birds can count up to 45 or 50 pecks ([Emmerton], [Hauser]).

It might be objected that Ayer achieves a standoff at best. One philosopher's *modus ponens* is another philosopher's *modus tollens*. Ayer argues that in fact we speak about contents, therefore not everything we speak about can be structure. Quine argues that we can only speak about structure, therefore we do not in fact speak about contents.[18]

It might also be objected that Ayer's argument begs the question. Without an explanation of how contents of sense-experiences can be communicated if there is a systematic reference shift across communicators, Ayer is postulating a magical solution that simply ignores the problem.

Both objections miss an important point. It is not a magical postulate, but the ordinary, pre-philosophical understanding that colors and countings are public. Russell's criticism relies on ordinary public counts of ordinary public sheep (compare: ordinary public colors). Russell defines number in logical terms which are abstractions from sense-experience. For Russell, we can learn number-terms only through empirical counting, even if we later restrict our use of those terms to pure arithmetic. More deeply, we can learn logical terms only through empirical observations of the presence or absence of things and their properties, even if we later restrict our use of those terms to pure logic. Thus "some," "all," and "none" are empirically learned just as much as "zero," "one," and "two" are. Russell is very clear that Frege's logicist sort of definition of number yields exactly the constants we need for ordinary counting [Russell 1927a, 4].[19]

A third objection is more difficult. Colors and numbers are not that similar. Ayer may be on strong ground for colors, which are paradigms

[18]A caveat: Quine admits prelinguistic quality spaces. See the previous note.

An argument indirectly supporting Ayer is that if everything is structure, then there is a vicious regress of structures, and nothing is anything. Russell gives such an argument in [Russell 1927b, 153, 319]. The 1918 Russell admits simples, but admits he might be wrong and that logical analysis might be endless [Russell 1918, 202] (see [Russell 1927a, 2–10], [Russell 1903, 145], [Russell 1948, 252]; [Russell 1959, 164–5]: 164–5).

[19]Russell finds that the universal negative statement "No ducks are in the box" is logically equivalent with the particular negative empirical statement "The property duckiness is not in the box" [Russell 1948, 132–3, 135–7, 504].

Kennedy says that for Peano, "mathematics can, and must, be based on experience" ([Kennedy 1974, 405], quoting Peano as saying, "The object studied by [mathematics] is given by the experimental sciences ....."). Rodríguez-Consuegra urges even more strongly that for Peano, numbers are given intuitively "by induction (abstraction) ... through ... experience," and "the axioms of arithmetic ... are empirical (being linguistic)" [Rodríguez-Consuegra 1991, 126–7]. Perhaps

of qualities in ordinary thought and language. If colors are not qualitative contents, what is? But there is something obviously very structural about the most ordinary arithmetic. Even Russell analyzes arithmetic as logical structure. This breathes new life into both earlier objections. Numbers do not appear to be qualitative as clearly as colors do; thus it might be a magical postulation to say that they are even contents. This objection is about the phenomenology of number, and here there could indeed be a ponens-tollens standoff. But we need reply only that if Russell is right, then to that extent numbers are constants.

4. Quine's Objections to Russell

Quine's objections to Russell on behalf of Peano are well known.

First, Quine says that Russell's own specific notion of *Anzahl*, or the notion of "n so-and-sos," is definable "without ever deciding what numbers are, apart from their fulfillment of arithmetic, simply as meaning that the so-and-sos are in one-one correspondence with the numbers up to n" ([Quine 1969, 44], citing [Quine 1963, § 11]). My reply is that this begs the question. Counting n so-and-sos by placing them in *ordinary* one-one correspondence with the *ordinary* numbers up to *ordinary* n simply ignores Russell's criticism.[20]

Second, Quine advocates odd counting as a compensatory adjustment that defends Peano against Russell:

> The condition upon all acceptable explications of number can be put ...: any progression ... will do nicely. Russell once held that a further condition had to be met, to the effect that there be a way of applying one's would-be numbers to the measurement of multiplicity: a way of saying that
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> (1) There are n objects x such that Fx.

then Peano himself might wish to apply Russell's criticism to Dedekind and Quine. Russell's comment is that since Peano's axioms fail to provide constant meanings for his undefined terms:

> His only method, therefore, is to say that at least one such constant meaning can be immediately perceived, but is not definable. This method is not logically unsound, but it is wholly different from the impossible abstraction which he suggests.
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> [Russell 1903, 127]

[20]See especially [Russell 1903, 114, 133, 309]. "Counting" is listed in the index.

> This, however, was a mistake: any progression *can* be fitted to that further condition. For, (1) can be paraphrased as saying that the numbers less than n admit of correlation with the objects x such that Fx .... (We have to count from 0 instead of 1 to *make* this come out right, but this is little to ask.)
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> [Quine 1960, 262–3] (my emphasis)

The key words are "can" and "make." Quine's point is really this: not every progression *does* fit Russell's condition, but we *can make* it fit, by making a compensatory adjustment to counting procedure. This is like using the bed of Procrustes to measure the length of travellers. That most travellers can be made to fit the bed only by stretching them on a rack or chopping off their heads or feet only shows that most travellers do not actually fit the bed. That most odd progressions can be made to fit Russell's condition only by making compensating adjustments to our counting procedure only shows that most progressions do not actually fit the condition.

How is it possible for Quine to *make* any progression fit Russell's condition? No Quinean compensating adjustment is a compensating adjustment *by accident*. We always know (i) that a certain term refers to the *wrong* referent, but we also know (ii) how to preserve the truths in question by making *another* term refer to a wrong referent as well. And we can know that only if we know the *right* referents. Even Quine has a home language in which he can identify oddities and compensatory adjustments, whatever they are [Quine 1969, 47–9] (see [Quine 1980–1, 20], [Quine 1986, 460]).

Third, Quine criticizes Russell's "more general point about application," i.e., about counting, by saying that since ostensions themselves can be multiply interpreted, "Always, if the structure is there, the applications will fall into place" [Quine 1969, 44]. Quine concludes that "any progression will serve as a version of number so long and only so long as we stick to one and the same progression. Arithmetic is, in this sense, all there is to number: there is no saying absolutely what the numbers are; there is only arithmetic" [Quine 1969, 45].[21] My reply is that this too begs the question. This is just the second criticism again. Quine says the applications "will fall into place" by themselves. But as we saw, counting does not fall into place in odd*, odd**, odd***,

[21]Quine cites [Benacerraf, 47–73]. Dummett criticizes Benacerraf by repeating Russell [Dummett 1991, Ch. 5].

or odd###. To *make* counting fall into place, I introduced counting####. But counting still does not fall into place in odd*, odd**, odd***, or odd###.

To paraphrase Russell on denoting, there is no backward road from structure to content, and no backward road from pure arithmetic to its empirical applications. Quine agrees that we cannot squeeze content out of structure, and abandons content. But he cannot abandon empirical applications, and thus claims we can squeeze them out of structure. Insofar as empirical content and empirical applications are logically tied, Quine is inconsistent, and takes the backward road. Russell's famous thesis that "there is no backward road from denotations to meanings" [Russell 1918, 50] states the point more generally, since Russell analyzes only denotations as structures, and analyzes meanings in terms of particulars and universals with which we are acquainted. But even for the later Wittgenstein, applications in mathematics do not just fall into place. They only seem to after the game is learned because the game develops from them. Thus there is no backward road from pure arithmetic to ordinary counting. Quite the opposite. For Wittgenstein, it is the ordinary use that is given as unquestioned, and it is the necessary truth of pure arithmetic that is in question [Wittgenstein, I §§ 1–4 ff.]. And odd counting procedures are intelligible to us only as specifiable deviations from ordinary use (compare [Dummett 1966], [Stroud]).

Now, Quine defends explications by saying that followers of "Wittgenstein deplore them as departures from ordinary usage, failing to appreciate that it is precisely by showing how to circumvent the problematic parts of ordinary usage that we show the problems to be purely verbal" [Quine 1960, 261]. We might try to use this point to defend Peano against Russell. But this turns things upside down. It is not ordinary counts that are problematic, but the odd counts of Peano arithmetic.

Fourth, in light of Quine's "no entity without identity" thesis, it would be a deep objection to say that in odd####, the criterion for the identity of ducks has changed, since we are counting a pair of ducks as one duck when we count the first pair of ducks as one#### duck, and there is a logical tie between "the same duck" and "one duck," as shown in the expression "one and the same duck." My reply would be that in odd#### arithmetic, we are counting not ducks, but ducks####. The criterion for the identity of ducks has not changed. Instead, we are using the criterion for the identity of ducks####, and the logical tie is between "the same duck####" and "one duck####." But again, we do not have to speak of counting a pair of ducks, or of skipping over the first duck, or even of counting

ducks####. We are simply using ordinary one-one correspondence to count the ducks, and subtracting the number one from the count. The numbers and the counting procedure are odd####, but not the ducks or the criterion for duck identity. That may seem to drive an intolerable wedge between identity and the number one, but really it does not. After all, the logical tie between "the same duck" and "one duck" exists only in ordinary arithmetic, and in odd arithmetics that have ordinary counting procedures. One would not expect the tie to exist in odd#### arithmetic, that is, to exist between identity and the number one####. More precisely, the ordinary tie exists in odd#### counting but only as a logical part of the counting procedure, the other part being to subtract one from the count; and these parts are discernible only metalinguistically, that is, from outside odd####.

In a totally odd arithmetic, the counting procedure stipulates what counts as counting the ducks. Even if the procedure requires that we must conceptualize the procedure as skipping over a duck, this is only from the perspective of ordinary counting. From within the perspective of the totally odd arithmetic, nothing is skipped over, since proper procedure is being followed. It is only in our description of the odd arithmetic from the perspective of ordinary arithmetic, i.e., only in our ordinary meta-language, that we may speak of the odd arithmetic as skipping over a duck.[22]

That odd#### counting is a form of counting is shown by its intertranslatability with ordinary counting. In fact, I defined it as a certain specific deviation from ordinary counting. And conversely, ordinary counting may be defined in terms of odd#### arithmetic. The ordinary count is definable as the odd#### count plus#### one####. This is analogous to odd geometries. It was initially questioned whether non-Euclidian geometries are "really" geometries as opposed to bizarre, merely verbal exercises. But that has not been seriously questioned since it was found that they are intertranslatable with Euclidian geometry. Their uses were given with their models within Euclidean space—positively or negatively curved planes. Likewise, odd counting has its uses within the model of ordinary counting. Its use for secret codes or encryptions of counts is obvious. More mundanely, whenever we ignore selected subtotals in doing inventory or refer to thirteen biscuits as a "baker's dozen," we are using odd arithmetics. As Wittgenstein would say, such ordinary language uses are all

[22]When Russell interprets Peano's numbers as the even numbers [Russell 1919, 7], there is no sense of skipping over anything. There are only even numbers, and no even numbers are skipped.

right.[23] But like non-Euclidean geometries before scientific uses were found for them, odd counts need not be practically useful in order to be intelligible.

Fifth, Quine is saying that no matter how we interpret Peano's undefined terms, 0 is 0, 1 is $s(0)$, 2 is $s(s(0))$, 3 is $s(s(s(0)))$, 4 is $s(s(s(s(0))))$, and 5 is $s(s(s(s(s(0)))))$.[24] Thus no matter what we interpret 5 as being, Peano ensures that 5 is the fifth successor after 0. For where an interpretation of a number term is called an element, if we put whatever elements we are counting into ordinary one-one correspondence with numbers, then however oddly we may have selected the elements, five elements will always correspond to five number terms.[25] I might add that while we can skip over ducks, by definition we cannot skip over elements. My reply is that this was never at issue, and Russell and I never denied it. In fact, it is the basis of Russell's criticism and of my extension of it. What is at issue is whether, on every interpretation that satisfies Peano's axioms, we will *say* there are five ducks *in* Peano arithmetic if and only if there are five ducks. That is, the issue is whether Peano arithmetic is an adequate *object language* for counting.

Russell is well aware that Peano's axiom (3)implies that each number has a successor different from that of any other number [Russell 1919, 6]. Russell is arguing that even though in Peano arithmetic there are five predecessors of 5, we cannot *say* that *in* Peano arithmetic, because whatever we say *in* Peano arithmetic is subject to infinitely many interpretations. For example, on the odd*** interpretation, when we say in Peano arithmetic, "There are five predecessors of 5," we mean there are *six* predecessors of *6*, which is true but not the intended content. And when we say "There are five number-expressions preceding the expression '5'," we mean there are *six* number-expressions preceding the expression "5," which is not even true.

To be adequate for counting, Peano arithmetic must be adequate as the object language *in* which counting is done. If we must resort to ordinary English as a metalanguage for saying *about* Peano arithmetic that it has five predecessors of 5, or for saying *about* the five expressions in Peano arithmetic "1," "2," "3," "4," and "5" that they can be put in ordinary one-one correspondence with five ducks, then Peano arithmetic fails to be a language *in* which counting can be

[23]Compare Wittgenstein [Wittgenstein, I #5, #139] on practical reasons for different or unusual mathematical uses.

[24]I use Sayward's notation [Sayward, 29].

[25]I attribute this objection to Torkel Franzén with thanks.

correctly—or even determinately—done. Thus the objection confuses the use and the mention of Peano arithmetic. Specifically, it overlooks that when we *use* Peano arithmetic to count things, the odd*** counts of the predecessors of a certain number, and even the odd*** counts of number-expressions—even of the number-expressions of odd*** arithmetic itself—will be just as wrong as odd*** counts of anything else.

The final two objections are implicit in Quine.

Sixth, one might object that surely we can use Peano numbers to count the Peano number-expressions correctly.[26] My reply is that this presupposes ordinary one-one correspondence. Using numbers to count their own names is no different from using them to count anything else. The odd#### counter must count oddly because of the odd#### counting procedure. It makes no difference whether the expressions are "1," "2," "3," "4," and "5" or whether we dress them up as "$s(0)$," "$s(s(0))$," "$s(s(s(0)))$," "$s(s(s(s(0))))$," and "$s(s(s(s(s(0)))))$." If we say there are "five" in odd***, we mean there are six. And if we want to mean there are five, in odd*** we have to say there are "four." If Peano arithmetic is adequate for counting, then why can it not even be used to count its own number-expressions?

Seventh, it would be a deeper objection to say that we should at least be able to use Peano numbers to count *themselves* correctly, even though there are no specific, determinate things that they are.[27] Surely Peano numbers must be in one-one correspondence with themselves, because every thing must be in one-one correspondence with itself. Surely every thing must be in one-one correspondence with itself because every thing is identical with itself. Thus being unable to use numbers to count themselves amounts to violating the axiom of self-identity, $(x)(x = x)$. Also, using numbers to count themselves is comparable to using the standard meter bar in Paris to measure itself. Namely, the result should be tautologically correct. My reply is once again that this presupposes ordinary one-one correspondence. It is only ordinary one-one correspondence which is logically tied to the axiom of self-identity so as to mandate the correct result when we use numbers to count themselves. Using numbers to count themselves is no different from using them to count anything else. The odd counter must use the odd counting procedure. If Peano arithmetic is adequate for counting, then why can it not even be used to count its own numbers?

[26] I take this to be part of Franzén's objection.

[27] I attribute this objection to Raymond Perkins with thanks.

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